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Hence, the rate of change of the strength of the mixture with respect to the volume of liquid added, is given by

$$\frac{ds}{dv} = \frac{s_1 - s}{v_0} \quad \text{or} \quad \frac{ds}{s_1 - s} = \frac{dv}{v_0}.$$

The value of $v$, the volume necessary to raise the strength from $s_0$ to $s$, is given by

$$\frac{v}{v_0} = \int_{s_0}^{s} \frac{ds}{s_1 - s} = -\log\frac{s_1 - s}{s_1 - s_0},$$

*i. e.*, by

$$s = s_1 - (s_1 - s_0)e^{-v/v_0}$$

as before.

February, 1919.

# RECENT PUBLICATIONS.

## REVIEWS.

### VEBLEN AND YOUNG'S PROJECTIVE GEOMETRY.

*Projective Geometry.* By O. VEBLEN and J. W. YOUNG. Boston: Ginn. 8vo. Vol. 1, 1910, reprinted in 1916. 10 + 344 pages; price $4.00. Vol. 2, 1918. 12 + 511 pages; price, $5.00.

To dispel at the outset any misunderstanding concerning the following comment and criticism of the reviewer it must be stated that, in his opinion, Veblen and Young's Projective Geometry is a very scholarly and profound treatise on the axiomatic foundations of projective geometry, its classification, and the main body of projective propositions, or theorems. As such it undoubtedly reflects credit upon American scientific scholarship.

All through the two volumes one is constantly impressed with the fact that the axiomatic foundations are extremely important, and that the chain of principal theorems is merely a byproduct of axiomatic, and not the main object of, scientific research in general. This impression may of course be due to the fact that there are two irreconcilable classes of scientific minds: idealists and realists, just as in philosophy.

For this reason it is as a rule useless to argue a case belonging to the philosophy, or psychology, of this domain between representatives of the two different classes. The idealist maintains that the mind is free to create anything it pleases, without reference to any sense-perception whatsoever; mathematics in its foundation may be based upon pure logic alone. It is the purely logical process which, for him, is the essential thing. The realist, on the other hand, holds fast to the famous doctrine: "Nihil est in intellectu quod non prius fuerit in sensu." According to him our entire knowledge, fundamentally, may be traced back to the impressions made upon our mind by the influx through the senses. The choice of a set of axioms for a reasonable, not only theoretically possible, scientific doctrine, must be dictated by the desire to create a system whose possible interpretations and applications do not collide in essential parts with our empirical knowledge of the external world. For him the logical structure is erected for the purpose of housing mathematical content conveniently and elegantly. That the foundations of such a system cannot be established without intuitional direction, at least in parts, seems clear to the realist. Poincaré,[1] for example, maintains that some of Hilbert's axioms of order are not free creations of the mind, on the contrary that they bear the mark of intuitional truth, that they are intuitional propositions. According to Poincaré, who, as is well known, has meditated profoundly upon analysis situs, the axiomatic foundations of this important branch of mathematical science cannot be laid entirely without some direction by intuition.

The idealist and realist agree, of course, that after the establishment of a set of axioms the erection of the scientific system, or systems, for which the set is valid, must proceed rigorously according to the demands of formal logic.

But here is another difficulty. How are we going to prove that a system is consistent without the aid of some previously known consistent system? Or are we supposed to go on building the structure and to say that it is consistent as long as it does not collapse; *i. e.*, as long as no contradictions turn up? In this connection see the foot-note on page 9 of volume 1.

After this short digression it would seem apparent that the set of assumptions $A$, $E$, $K$ for a real projective space, for example, in spite of the author's assertion, is not "very arbitrary," but is chosen with the purpose of creating a real projective geometry, as it is customarily understood.

Volume I is the product of the joint labor of Veblen and Young, while for Volume 2 Veblen alone holds himself responsible. According to some statements in the preface the authors realize that from the standpoint of pedagogy the contents might possibly be treated and may be arranged differently. "The second volume has been arranged so that one may pass on a first reading from the end of chapter VI, volume 1, to the beginning of volume 2. The later chapters of volume 1 may well be read in connection with the part of volume 2 from chapter V onward."

On page 95, volume 1, the following *provisional assumption of projectivity* is made:

P. *If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.* This is, of course, von Staudt's theorem, to which more explicit reference should be made. Based upon this the fundamental theorem of projective geometry, printed on the same page, then follows easily: *If* 1, 2, 3, 4 *are any four elements of a one-dimensional primitive form, and* $1'$, $2'$, $3'$ *are any of three elements of another or the same one-dimensional primi-*

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[1] *Dernières Pensées*, 1917, pp. 92–97. See in this connection also recent discussions in the *Monist* (January, 1919) by Richard A. Arms on "the relation of logic to mathematics" and by V. F. Lenzen on "independence proofs and the theory of implication."

*tive form, then for any projectivities giving* $1234 \barwedge 1'2'3'4'$ *and* $1234 \barwedge 1'2'3'4_1'$, *we have* $4' = 4_1'$.

The foregoing provisional assumption contains the key to the whole situation, and its rigorous proof from a set of axioms of connection, order, and continuity has required the successive efforts of von Staudt, Klein, Lüroth, Zeuthen, Darboux, and others.

The discussion of the foundations is again taken up in chapter I of volume 2, with the object of obtaining a classification of various projective geometries. In this the symbols denoting the various assumptions have the following meaning:

$A$: Alignment,
$E$: Extension,
$P$: Projectivity,
$K$: A geometric number system is isomorphic with the real number system of analysis,
$H$: If any harmonic sequence exists, not every one contains only a finite number of points,
$H_0$: The diagonal points of a complete quadrangle are non-collinear.
$J$: A geometric number system is isomorphic with the complex number system of analysis,
$C$: Continuity,
$R$: On at least one line, if there is one, there is not more than one chain.
$\bar{R}$: On some line, $l$, not all points belong to the same chain.
$I$: Through a point $P$ of any chain $C$ of the line $l$, and any point $J$ on $l$ but not in $C$, there is not more than one chain of $l$ which has no other point than $P$ in common with $C$,
$\bar{H}$: If any harmonic sequence exists, at least one contains only a finite number of points.
$Q$: There is more than one net of rationality on a line.
$S$: Senses of ordered projective spaces. A space satisfying assumptions.
$A$, $E$ is a general projective space,
$A$, $E$, $P$ is a proper projective space,
$A$, $E$, $H$ is a non-modular projective space,
$A$, $E$, $\bar{H}$ is a modular projective space,
$A$, $E$, $S$ is an ordered projective space,
$A$, $E$, $\bar{H}$, $Q$ is a rational modular projective space,
$A$, $E$, $H$, $Q$ is a rational non-modular projective space,
$A$, $E$, $H$, $C$, $R$ or $A$, $E$, $K$ is a real projective space,
$A$, $E$, $H$, $C$, $\bar{R}$, $I$ or $A$, $E$, $J$ is a complex projective space.

*A set of assumptions is said to be categorical, if there is essentially only one system for which the assumptions are valid, i. e., if any two such systems may be made simply isomorphic,* page 6, volume 1. Thus $A$, $E$, $H$, $C$, $R$ form a categorical set.

This classification, the establishment of series of propositions in these various geometries, and the logistic theory leading to them, characterizes the whole treatise. It is very commendable that the authors have accomplished this task, which was not a very easy one, with marked success. Americans, and for that matter all geometers to whom the book is accessible, will be glad to refer students who desire more information on foundational methods in geometry to Veblen and Young's competent treatment of this branch of mathematics.

As far as the didactic treatment is concerned, the reviewer expresses the opinion that in a projective geometry a more central, or prominent position should be given to the fundamental theorem, and it would have been more consistent to have incorporated the first chapter of volume 2 in volume 1 before the fundamental theorem was reached. Enriques's *Vorlesungen über projektive Geometrie* offers an excellent example of such a treatment. Also Del Pezzo's *Principi di Geometria Projettiva* follows a similar plan with much success.

Although a logistic treatment of geometry should be possible without figures of any kind, Veblen and Young were wise not to exclude the mechanical auxiliary of graphic illustrations, as is done by Del Pezzo. I ask the question whether anybody without preliminary knowledge, or the analytic equivalent, could understand Del Pezzo's book at all without drawing most of the figures to which he refers? Again, who is able to write a treatise on projective geometry without either the knowledge of the analytic equivalent, or the drawing of at least some of the figures as a purely mechanical auxiliary? If an author makes use of figures in preparing a treatise, why should those figures be deleted on publication? But the point is really this: Since a logistic system starts with a set of *undefined elements*, and a set of assumptions between them, independent of intuition, the use of figures in such a system should be justified. What relation connects the points and lines of the logistic system with the physical points and lines of the figure and the sheet of paper upon which they are drawn? It seems to me that this difficulty can only be overcome by a set of *axioms of construction* admitting the possibility of actual graphic representation, isomorphic with the logistic system.

As to the content of the two volumes in general, it would be impossible to give a detailed account of it within the narrow space of this review. I shall therefore confine myself to some features of particular interest.

The fourteen chapters of the first volume deal with theorems of alignment and the principle of duality; projection, section, perspectivity, elementary configurations; projectivities of the primitive geometric forms of one, two, and three dimensions; harmonic constructions and the fundamental theorem of projective geometry; conic section; algebra of points and one-dimensional coördinate systems; coördinate systems in two and three-dimensional forms; geometric constructions, invariants; projective transformations of two-dimensional forms; families of lines.

As has been pointed out before, von Staudt's theorem, and thereby implicitly the fundamental theorem, is assumed for a proper projective geometry and receives its conclusive aspect at the beginning of the second volume. Thus the theory of conics in chapter 5, for example, is based upon this assumption $P$

(and, of course, $A$, $E$, $H_0$). An account of the historic development, or of the evolution, of the fundamental theorem, and a statement as to the motive for making one particular choice of assumptions in preference to another, would have greatly added to the clearness of the exposition. The reviewer shares with Poincaré the opinion that the beginner should be given the reason why a particular choice of axioms is made, because it seems to be evident that the choice cannot be "very arbitrary" if the geometer has in view the establishment of a certain system of geometry. Moreover the student undoubtedly shows increased interest if he is familiarized with the historic roots of the science which he undertakes to study. The authors, in their treatise at least, show little interest in the historic side of the subject. Historic and some other references given are rather meager and partly insufficient. Thus, in a foot-note on page 109 we are told that Jakob Steiner (1796–1863) was the first to define the first four one-dimensional forms of the second degree by projective pencils. Why not add when and where?, *i. e.*, in the *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*, etc., which appeared in 1832.

In the definition of homogeneous coördinates in the plane it is important to state explicitly that the ratios of any two coördinates of a point $(x_1, x_2, x_3)$, or a line $(u_1, u_2, u_3)$, may be represented by cross-ratios, which are not changed by collineations. This is the reason why they are frequently called projective coordinates, as introduced into geometry by Fiedler.[1] It would be more natural to incorporate exercise 1, on page 179, into theorem 3 on the same page. Choosing the same triangle of reference for point and line coördinates, and a unit line $(\rho u_1 = 1,\ \rho u_2 = 1,\ \rho u_3 = 1)$ independent of the unit point $(\delta x_1 = 1,\ \delta x_2 = 1,\ \delta x_3 = 1)$, the condition for coincidence for a point $(x_1, x_2, x_3)$ and a line $(u_1, u_2, u_3)$ is $k_1u_1x_1 + k_2u_2x_2 + k_3u_3x_3 = 0$, where $k_1$, $k_2$, $k_3$ are three constants which do not vanish simultaneously. When the unit-line is the triangle-polar of the unit-point, then $k_1 = k_2 = k_3$ and the necessary and sufficient condition for coincidence reduces to $u_1x_1 + u_2x_2 + u_3x_3 = 0$. A similar remark may be made with respect to projective coördinates in space of three dimensions.

In Chapter IX on geometric constructions is introduced the term *geometric operation*, by which the authors give expression to the feeling that something should be done in passing from an axiomatic to a concrete system. This is precisely the point which I had in view when I wrote of the necessity of suitable axioms of construction. The statement, on pp. 236–237, that there is no familiar mechanical device for drawing lines and planes in space is not correct. Just as one can manufacture a ruler, one can manufacture a *planoid*, *i. e.*, a plane surface, by means of which planes in space may be drawn, or mechanically fixed. Moreover we have Koenig's planigraph,[2] a linkage, by which a plane may be described in analogy with the description of a straight line by Peaucellier's inversor.

On account of the importance of the subject it would have been well if the classification of collineations and their group-properties in Chapter X had received a more extensive treatment. The same is true of the discussion of pairs of conics on pp. 287–293. The parametrically linear form which is of importance in this discussion is the pencil of conics and the classification of such pencils. We also miss the statement that, in general, the discriminant of a pencil of conics $\phi + \lambda\psi = 0$, set equal to zero, is a cubic in $\lambda$, so that, in general, there are three degenerate conics in a pencil.

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[1] *Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich*, Vol. 15, p. 152f. (1870).

[2] *Leçons de Cinématique*, pp. 295–297.

The last chapter on families of lines deals with the regulus, its principal properties, and the figures of lines depending on these properties. It is important to know that, from a certain point of view, the problem of projectivity is closely related with the theory of the regulus. In arts. 109 and 110 a very brief treatment of Plücker's line coördinates is given. The fact ought to be mentioned that the coördinates

$$\rho_{ik} = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}, \qquad \phi_{ik} = \begin{vmatrix} u_i & v_k \\ v_i & u_k \end{vmatrix}$$

are commonly known as *homogeneous radial*, and *axial coördinates* of the line, respectively. Considering their great importance in other fields, the linear complex and in particular the null system, and line-congruences deserve fuller consideration than is given to them in the book under review. The authors realized, of course, that in their treatment space did not permit of too extensive accounts of even important specific geometric systems.

The scope of the second volume, written by Veblen, is very clearly stated in the preface:

"We have in mind two principles for the classification of any theorem of geometry: (*a*) the axiomatic basis, or bases, from which it can be derived, or, in other words, the class of spaces in which it can be valid; and (*b*) the group to which it belongs in a given space. . . . Having fixed attention on any particular space, we have a set of groups of transformations to each of which belongs its geometry. For example, in the complex projective plane we find among others, (1) the group of all-continuous one-to-one reciprocal transformations (analysis situs), [why add the word reciprocal?], (2) the group of birational transformations (algebraic geometry), (3) the projective group, (4) the group of non-Euclidean geometry, (5) a sequence of groups connected with Euclidean geometry. . . . The two principles of classification, (*a*) and (*b*), give rise to a double sequence of geometries, most of which are of consequence in present day mathematics. It is the purpose of this book [volume 2] to give an elementary account of the foundations and interrelations of the more important of these geometries (with the notable exception of (2))." Veblen suggests the desirability of other books taking account of this logical structure, but dealing with particular types of figures. I take it for granted that the author does not want to intimate by this that there are no treatises on geometry in existence which are based upon a logical structure. The projective geometry of that three dimensional space which is simply isomorphic with analytic space containing as points all quadruples

$(x_1, x_2, x_3, x_3)$ of real or complex numbers, of which not all four ever vanish simultaneously, and as planes all linear equations $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ between the coördinates of these points, with the $a$'s as arbitrary constants from the same number system, for example, is well established according to the requirements of a logical system.

The second volume contains nine chapters on foundations, elementary theorems on order, the affine group in the plane, Euclidean plane geometry, ordinal and metric properties of conics, inversion geometry and related topics, affine and Euclidean geometry of three dimensions, non-Euclidean geometries, theorems on sense and separation.

Affinity is given a prominent place in Chapters III and VII, which treat of the affine groups in the plane and in space, respectively. As the systems of affinity in the plane as well as in space express properties invariant under a given group of transformations, they may be called geometries in the sense of Klein and properly specified by the word affine. Considering the fact that affinity is recognized as a geometry of marked importance in a number of recent investigations, the 112 pages reserved for the two chapters prove indeed that the author is aware of this fact. Chapter III is based entirely on assumptions $A$, $E$, $P$, $H_0$. In fact a large number of theorems depend only on $A$, $E$, $H_0$. The affine group of three dimensions, of which the whole affine plane geometry is a part, in a nonhomogeneous coördinate system in which $\pi_\infty$ is the singular plane, consists of the set of all projectivities of the form

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{10}, \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_{20}, \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_{30}, \end{aligned}$$

where

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

The set of points not on $\pi_\infty$ is called a *Euclidean space* and $\pi_\infty$ is called the *plane at infinity* of this space, which is invariant in the affine group of three dimensions. The corresponding geometry is called *affine geometry*, or simply *affinity of three dimensions*.

To non-Euclidean geometries in Chapter VIII only 31 pages are given. These are approached in the most natural manner by means of *the absolute* as introduced by Cayley and clearly brought out by Klein in its connection with non-Euclidean geometries. I wonder what our friend Halsted will say if he finds that in the references to the history and exposition of parts of non-Euclidean geometry on page 362, his name is omitted?

In comparison to the size of this chapter, the one following on theorems on sense and separation with 116 pages, more than one fifth of the whole book, seems exceptionally large. But this chapter contains much of Veblen's own investigations on this branch of analysis situs.

A very commendable feature of the whole treatise is the large number of exercises, which are either original, or selected from many well-known standard works and monographs. Among them are problems of considerable importance in real projective geometry, which by some authors are included in the main subject matter.

The mechanical make up of the two books is excellent, with the exception of many figures, which, from the standpoint of a connoisseur of graphic arts, are by no means the product of expert draftsmen. Pleasing figures, however, are as much to be desired as pleasing type. In this respect the second volume shows a decided improvement over the first.

May be it is a whim of an æsthetic crank if he misses the same kind of imprint on the backs of the two volumes. Why is that of the first volume gilt, that of the second volume black? But we shall not insist further upon such trivialities, and close this review by thinking with Voltaire: Le secret d'ennuyer est celui de tout dire.

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*An Elementary Treatise on Curve Tracing.* By P. FROST. Fourth edition, revised by R. J. T. BELL. London, Macmillan, 1918. 8vo. 16 + 210 pp. + 17 plates. Price 12s. 6d.

The first edition of this work appeared in 1872 and reprints were issued as second and third editions in 1892 and 1911 respectively. Concerning the present edition (which is the first revision of the original work) Mr. Bell states that it does not differ from the previous one except in places where alterations were necessary to remove ambiguities or to correct mistakes in analysis and diagrams. A useful classified list of the curves discussed has been added on pages 203–208, and the typography of the new edition is much better than that of the earlier one. Except for 7 types of curves whose equations involve trigonometric functions, all curves analyzed have rational algebraic equations; these include 23 cubics, 69 quartics, 31 quintics, 18 sextics, 9 curves of the seventh degree and 2 of the eighth. As the calculus is nowhere employed in the discussion great ingenuity is displayed in the analyses, and the work as a whole is exceedingly interesting.

In his preface of 1872 (reprinted in the present edition) Frost wrote that it would be difficult to find another subject

> "which, with a very limited extent of reading, combines, to the same extent, so many valuable hints of methods of calculations to be employed hereafter, with so much pleasure in its use.
>
> For example, the subject of Graphical Calculations is coming more into use every day, and is applied with success to many difficult problems in Statics, Engineering and Crystallography; hints of this the student will find in the practical solution of divers questions and in the determination of the number of their real roots, which are obtained by graphical methods with great facility.
>
> Again, the methods of successive approximations which are employed in Optics and Astronomy are illustrated in the process of finding asymptotes and approximations to the forms of curves at a finite distance.